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Bending of circular-section bonded rubber blocks

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Abstract

Convenient exact closed-form expressions are derived for calculating the bending stiffness of and stresses within loaded cylindrical bonded rubber blocks of circular cross-section. The particular solutions for simple bending, cantilever loading and apparent shear situations are deduced and studied in detail. The shapes of the deformed profiles are discussed and confirmation is provided that the previously adopted assumption of parabolic profiles of the deformed lateral curved surface is only valid for blocks of very small aspect ratio. In simple bending a relationship which is more realistic than those hitherto suggested is derived for the couple required to maintain a specified rotation of the loaded end of the block. In apparent shear an exact expression for the ratio of the true to the apparent shear modulus is derived, and compared with the experimental data. An improved approximate relation is deduced.

Keywords: Rubber blocks; Bending; Circular-section

1. Introduction

The properties of rubber mountings bonded to rigid metallic end plates have been exploited widely for many years in a variety of engineering components. It is therefore extremely important to be able to predict their deformation and stiffness under specified applied loads.

An analysis is presented here of a rubber block of right-circular cross-section with one end maintained in a fixed position while the other end is subjected to a couple and shear force. In general this causes the block to bend, with its loaded surface tilting and deflecting laterally.

The fundamental problem is formulated in Section 2, and then, in Section 3, an exact analytical solution to the governing equations under a comprehensive loading system is derived. This enables detailed discussions to be undertaken in Sections 4–6, respectively, of situations corresponding to simple bending, cantilever loading and apparent shear.

Rivlin and Saunders (1949) investigated experimentally the effects of bending and shear on various cylindrical shear mountings. They attempted to determine the modulus of rigidity of the rubber, and

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suggested a theoretical approximate formula for the ratio, μ/μ_a, of the true shear modulus to the apparent shear modulus with which they compared their results. Their treatment was critically reconsidered by Gent and Meinecke (1970), who proposed alternative expressions for the bending stiffness factors in apparent shear for various cross-sections. More recently, Tsai and Lee (1999) developed a pressure approach to determine the tilting stiffness of an elastic layer bonded between rigid plates.

These previous analyses rely on the kinematic assumptions that not only do the cross-sectional planes remain planar but that the lines initially normal to the bonding plates become parabolic under deformation.

It is specifically confirmed in the present paper in Sections 4.3 and 6.3 that the assumption of a parabolic profile is in general invalid. From the exact general solution given here, with this assumption relaxed, an expression is derived for the shear moduli ratio in apparent shear which enables an improved approximation, $\mu/\mu_a^{\text{approx}}$, to be deduced. Comparisons are given with the experimental data of Rivlin and Saunders (1949). Similarly, in simple bending a more realistic relationship between the couple required to maintain a specified rotation of the loaded end of the block is deduced.

2. Theoretical formulation

Consider a rubber block of right-circular cross-section, with radius a and axial height h. A rectangular Cartesian coordinate system (x, y, z) is defined relative to an origin O at the centre of one of its end faces with Oz along the axis of the block, as shown in Fig. 1. The cylindrical polar coordinates (r, θ, z) of a point P within the block are related to its Cartesian coordinates by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The rubber is bonded to two rigid end plates at $z = 0$ and h that prevent all distortions of its plane end surfaces with the end at $z = 0$ held in a fixed position. The other end at $z = h$ is subjected to a shear force of magnitude F to the face in the $-$Ox direction together with a moment of magnitude $M - Fh$ in the negative direction of the y-axis. This will tilt the face $z = h$ through a small angle of rotation ϕ about the y-axis and laterally displace it a small distance d in the $-$Ox direction.

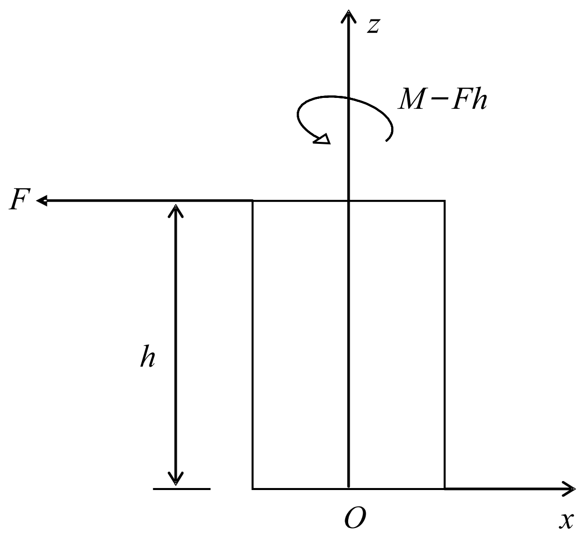

Fig. 1. Undeformed cross-section of the block through the $y = 0$ plane.

It is assumed throughout that the rubber is isotropic, homogeneous and incompressible, and that the displacement gradients are sufficiently small for the classical linear theory of elasticity to be applicable. The radial, tangential and axial components of the displacement of the point P are denoted by u_r, u_θ and u_z, respectively, and the cylindrical strain and stress components by ε_{ij} and σ_{ij}, where i, $j = r$, θ or z, in the usual notation.

The strain–displacement gradients relations and the constitutive equations can be written as

$$\begin{aligned}&\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right),\\ &\varepsilon_{rz} = \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \quad \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right),\end{aligned} \tag{1}$$

$$\begin{aligned}&\varepsilon_{rr} = \frac{1}{3\mu}\left[\sigma_{rr} - \frac{1}{2}(\sigma_{\theta\theta} + \sigma_{zz})\right], \quad \varepsilon_{\theta\theta} = \frac{1}{3\mu}\left[\sigma_{\theta\theta} - \frac{1}{2}(\sigma_{rr} + \sigma_{zz})\right],\\ &\varepsilon_{zz} = \frac{1}{3\mu}\left[\sigma_{zz} - \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})\right], \quad \sigma_{r\theta} = \sigma_{\theta r} = 2\mu\varepsilon_{r\theta}, \quad \sigma_{rz} = \sigma_{zr} = 2\mu\varepsilon_{rz}, \quad \sigma_{\theta z} = \sigma_{z\theta} = 2\mu\varepsilon_{\theta z},\end{aligned} \tag{2}$$

where μ is the shear modulus. The assumption of incompressibility implies that, for small strains,

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = 0 \tag{3}$$

and the equilibrium equations which must be fulfilled in the radial and tangential directions (Spencer, 1980, Eq. (11.39)) are

$$\begin{aligned}&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0,\\ &\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r_\theta}}{r} = 0.\end{aligned} \tag{4}$$

3. Solution for a general loading

Expressions for the angular rotation and lateral deflection of the bonded end of the block at $z = h$ arising from the bending are first derived for a general loading situation.

During bending it is assumed that plane right-circular cross-sections remain plane whilst rotating through an angle $\alpha(z)$ from the $z =$ constant planes. Representations for the corresponding displacement components at the general point P in the rubber are therefore sought in the forms

$$u_r = U\cos\theta, \quad u_\theta = V\sin\theta, \quad u_z = \alpha r\cos\theta, \tag{5}$$

with the functions U and V depending upon r and z.

By substituting these into the incompressibility condition (3), using Eq. (1), it follows that

$$V = -\left(U + r\frac{\partial U}{\partial r} + r^2\frac{\mathrm{d}\alpha}{\mathrm{d}z}\right). \tag{6}$$

The shear and normal stress components can be expressed in terms of α and U, using Eqs. (1), (2), (5), and (6), as

$$\begin{aligned}\sigma_{r\theta} &= -\mu\left(r\frac{d\alpha}{dz}+\frac{\partial U}{\partial r}+r\frac{\partial^2 U}{\partial r^2}\right)\sin\theta,\\ \sigma_{rz} &= \mu\left(\alpha+\frac{\partial U}{\partial z}\right)\cos\theta,\\ \sigma_{\theta z} &= -\mu\left(\alpha+r^2\frac{d^2\alpha}{dz^2}+\frac{\partial U}{\partial z}+r\frac{\partial^2 U}{\partial r\partial z}\right)\sin\theta,\end{aligned} \tag{7}$$

and

$$\begin{aligned}\sigma_{zz} &= \sigma_{rr}+2\mu\left(r\frac{d\alpha}{dz}-\frac{\partial U}{\partial r}\right)\cos\theta,\\ \sigma_{\theta\theta} &= \sigma_{rr}-2\mu\left(r\frac{d\alpha}{dz}+2\frac{\partial U}{\partial r}\right)\cos\theta.\end{aligned} \tag{8}$$

The equilibrium equations (4) then yield the system

$$\begin{aligned}\frac{\partial\sigma_{rr}}{\partial r} &= -\mu\left(2\frac{d\alpha}{dz}+\frac{3}{r}\frac{\partial U}{\partial r}-\frac{\partial^2 U}{\partial r^2}+\frac{\partial^2 U}{\partial z^2}\right)\cos\theta,\\ \frac{\partial\sigma_{rr}}{\partial\theta} &= \mu\left(2r\frac{d\alpha}{dz}+r^3\frac{d^3\alpha}{dz^3}-2\frac{\partial U}{\partial r}+4r\frac{\partial^2 U}{\partial r^2}+r^2\frac{\partial^3 U}{\partial r^3}+r\frac{\partial^2 U}{\partial z^2}+r^2\frac{\partial^3 U}{\partial r\partial z^2}\right)\sin\theta,\end{aligned} \tag{9}$$

which is to be solved subject to the appropriate boundary conditions.

Elimination of σ_{rr} between Eq. (9) shows that α and U are related through the differential equation

$$3\frac{d^3\alpha}{dz^3}+\frac{\partial^4 U}{\partial r^4}+\frac{6}{r}\frac{\partial^3 U}{\partial r^3}+\frac{3}{r^2}\frac{\partial^2 U}{\partial r^2}-\frac{3}{r^3}\frac{\partial U}{\partial r}+\frac{\partial^4 U}{\partial r^2\partial z^2}+\frac{3}{r}\frac{\partial^3 U}{\partial r\partial z^2}=0. \tag{10}$$

It can be shown that Eq. (10) has an exact solution, which is finite for all values of z at $r=0$, given by

$$U = 3\int\alpha\,dz-\frac{3}{8}r^2\frac{d\alpha}{dz}+Z, \tag{11}$$

with $Z(z)$ an arbitrary function of z, which can be used to express the system (9) solely in terms of α and Z. Hence, by direct integration and imposition of the boundary condition that

$$\sigma_{rr}=0 \quad \text{at } r=a \quad \text{for all } \theta \text{ and } z, \tag{12}$$

the Eqs. (9) are satisfied by

$$\sigma_{rr} = -\frac{\mu r}{8}(a^2-r^2)\frac{d^3\alpha}{dz^3}\cos\theta \tag{13}$$

with

$$\frac{d^2 Z}{dz^2}=\frac{a^2}{8}\frac{d^3\alpha}{dz^3}-\frac{7}{2}\frac{d\alpha}{dz}. \tag{14}$$

Integrating Eq. (14) twice with respect to z gives an expression for Z which after substitution into Eq. (11) yields

$$U=\frac{(a^2-3r^2)}{8}\frac{d\alpha}{dz}-\frac{1}{2}\int\alpha\,dz+kz+k_1, \tag{15}$$

where k and k_1 are arbitrary constants.

An expression for the normal stress component σ_{zz} can now be given by using Eqs. (13) and (15) with the relationship $(8)_1$ in the form

$$\sigma_{zz} = \mu\left[\frac{7}{2}r\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{r}{8}(a^2 - r^2)\frac{d^3\alpha}{\mathrm{d}z^3}\right]\cos\theta. \tag{16}$$

The as yet unknown angle of rotation $\alpha(z)$ which appears in the expressions (13), (15), and (16) must now be determined. The normal stress component acting at all points on a $z =$ constant plane combine to be equivalent to the imposed couple of magnitude $M - Fz$ in the negative direction of the y-axis, so that

$$M - Fz = \int_0^{2\pi}\int_0^a r^2\sigma_{zz}\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta. \tag{17}$$

Evaluating this using Eq. (16) leads to the differential equation governing α as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{a^2}{84}\frac{\mathrm{d}^3\alpha}{\mathrm{d}z^3} = \frac{8}{7\mu\pi a^4}(M - Fz). \tag{18}$$

Its general solution can be written as

$$\alpha(z) = c_1\cosh\omega z + c_2\sinh\omega z + \frac{8z}{7\mu\pi a^4}\left(M - \frac{Fz}{2}\right) + c_3, \tag{19}$$

with c_1, c_2 and c_3 being arbitrary constants, and

$$\omega^2 = \frac{84}{a^2}. \tag{20}$$

The expressions (15), (19), and (20) which provide solutions to the differential equation (10) can be combined to give

$$U(r,z) = \frac{10}{\omega}\left(1 - \frac{63}{20}\frac{r^2}{a^2}\right)(c_1\sinh\omega z + c_2\cosh\omega z) + \frac{(a^2 - 3r^2)}{7\mu\pi a^4}(M - Fz) - \frac{2z^2}{7\mu\pi a^4}\left(M - \frac{Fz}{3}\right) - \frac{c_3}{2}z + kz + k_1. \tag{21}$$

The constants c_1, c_2, c_3 and k_1 can be determined to fulfil the boundary conditions imposed at the bonded ends of the block. Since the end at $z = 0$ is regarded as fixed,

$$\alpha(0) = 0 \tag{22}$$

$$U(r,0) = 0, \quad \text{for all } r, \tag{23}$$

while at $z = h$ there is no movement in the Oy direction, the required distance moved laterally by the end due to bending alone in the $-$Ox direction is d_b and its sought rotation is ϕ, so that

$$\left.\begin{aligned} u_r\sin\theta + u_\theta\cos\theta &= 0 \\ u_r\cos\theta - u_\theta\sin\theta &= -d_b \end{aligned}\right\} \text{ at } z = h \text{ for all } r, \tag{24}$$

$$\alpha(h) = \phi. \tag{25}$$

Subjecting the solutions (19) and (21) to the boundary conditions (22) and (23) and utilizing the relations (24) and (25) gives

$$c_1 = -c_3 = \frac{2\omega}{147\mu\pi a^2}\left(M\tanh\frac{\omega h}{2} + Fh\,\mathrm{cosech}\,\omega h\right), \tag{26}$$

$$c_2 = 2\omega k_1 = -\frac{2\omega M}{147\mu\pi a^2} \tag{27}$$

and leads to the representations

$$\phi = \frac{8h}{7\mu\pi a^4}\left(M - \frac{Fh}{2}\right)\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right), \tag{28}$$

$$d_b = \frac{1}{147\mu\pi a^2}\left[Fh + 42\left(\frac{h}{a}\right)^2\left(M - \frac{Fh}{3}\right) - \omega h\left(M\tanh\frac{\omega h}{2} + Fh\,\mathrm{cosech}\,\omega h\right)\right] - kh. \tag{29}$$

These give the angular rotation and lateral deflection of the end of the block at $z = h$ due to bending.

With the end of the block at $z = 0$ considered fixed, the components of the displacement perpendicular to the z-axis due to shear alone of points on a $z =$ constant plane within the block are assumed to be given by Eqs. $(5)_1$ and $(5)_2$ with

$$U = -\frac{Fz}{\mu\pi a^2} \tag{30}$$

and the deflection, d_s, of the end at $z = h$ in the $-Ox$ direction is thus

$$d_s = \frac{Fh}{\mu\pi a^2}. \tag{31}$$

This must be added to d_b to obtain the total lateral deflection of the bonded end at $z = h$.

The value of the remaining unknown constant, k, in Eq. (29) depends upon the values chosen for F and M. Situations in which $F = 0$, $M = Fh$ and $M = Fh/2$, which correspond to simple bending, cantilever loading and apparent shear, respectively, are considered in Sections 4–6.

4. Simple bending

If flexure of the block is created by the application of a pure couple M without an applied shearing force the block is often said to suffer "simple bending".

4.1. Angular rotation and lateral deflection

When $F = 0$, it follows from the expression (19) for α that the shape of the deformed block is symmetrical in the plane of bending about the displaced plane originally given by $z = h/2$, in the sense that

$$\alpha\left(\frac{h}{2} + z_1\right) - \alpha\left(\frac{h}{2} + z_2\right) = \alpha\left(\frac{h}{2} - z_2\right) - \alpha\left(\frac{h}{2} - z_1\right)$$

for $0 \leqslant z_1, z_2 \leqslant h/2$. At the point of intersection, where $\theta = 0$, of the plane of symmetry with the profile, the angle between this plane and the plane tangential to the surface is a right angle, and thus

$$\alpha\left(\frac{h}{2}\right) = \frac{\phi}{2} = \left(-\frac{\partial U}{\partial z}\right)_{z=\frac{h}{2}}. \tag{32}$$

From Eqs. (28) and (21), the requirement (32) yields

$$k = -\frac{2hM}{7\mu\pi a^4}\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right) \tag{33}$$

and hence, when $F = 0$, explicit representations for the lateral deflection, d, of the end $z = h$ (which equals d_b since there is no deflection due to shear) and ϕ can be written as

$$d = \frac{4h^2M}{7\mu\pi a^4}\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right), \tag{34}$$

$$\phi = \frac{8hM}{7\mu\pi a^4}\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right). \tag{35}$$

It is interesting to note from Eqs. (34) and (35) that in this case $d = h\phi/2$ for all values of h and a, which is the same relationship as holds in the classical theory of bending.

However, the classical expression giving the angle of rotation, ϕ, for a long beam is

$$\phi = \frac{hM}{EI}, \tag{36}$$

where $E = 3\mu$ is the Young's modulus of the rubber and $I = \pi a^4/4$ is the second moment of area of the circular cross-section about the y-axis. Here when h is large the result (35) is approximated by

$$\phi \approx \frac{8hM}{7\mu\pi a^4}, \tag{37}$$

which can be written analogously to Eq. (36) in terms of an effective second moment of area, I_e, as

$$\phi \approx \frac{hM}{EI_e}, \tag{38}$$

with

$$I_e = \frac{7\pi a^4}{24}. \tag{39}$$

The physical interpretations of the results for a beam are often quoted with respect to the effects upon its flexural rigidity or bending stiffness, B, which is defined in terms of its Young's modulus, E, and its second moment of area, I, as

$$B = EI \equiv \frac{hM}{\phi}. \tag{40}$$

This is analogous in many ways to the definitions of the radial and tilting stiffnesses of a cylindrical rubber bush mounting as analyzed by Horton et al. (2000a,b). It is thus clear that here the bonding of the ends of the rubber block to rigid plates stiffens it by a factor of $7/6$ $(= I_e/I)$ when h is large.

On the other hand now suppose that h is small in comparison with a. In an approximate theoretical treatment of bonded rubber blocks, Gent and Meinecke (1970, Eq. (8) and Table 2) suggested that the couple M required to maintain a rotation ϕ of one bonded end is given (in the present notation) by

$$M = \frac{E\pi a^4\phi}{4h}\left(1 + \frac{a^2}{6h^2}\right) \tag{41}$$

and subsequently Tsai and Lee (1999, Eq. (53)) gave

$$M = \frac{E\pi a^4\phi}{4h}\left(\frac{7}{6} + \frac{a^2}{6h^2}\right). \tag{42}$$

By expanding the hyperbolic tangent using the series representation given by Abramowitz and Stegun (1965, Eq. (4.5.64)), it can be shown that when $h^2/a^2 < 5/42$ the expression (35) can be closely approximated (with an error of magnitude less than 0.54%) to give

$$M = \frac{E\pi a^4 \phi}{4h}\left(\frac{7}{5} + \frac{a^2}{6h^2}\right). \tag{43}$$

This is of the same form as, but more realistic than, the relationships (41) and (42), since they yield corresponding errors of 13.82% and 7.84%, respectively, when $h^2/a^2 = 5/42$.

4.2. Stresses

The stress components that are created within the rubber under simple bending of the block can be determined from Eqs. (13), (16), $(8)_2$, (15), (7), and (19) in the forms

$$\begin{aligned}
\sigma_{rr} &= \frac{12Mr}{\pi a^4}\left(1 - \frac{r^2}{a^2}\right)\operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right)\cos\theta, \\
\sigma_{\theta\theta} &= \frac{8Mr}{7\pi a^4}\left[1 + \frac{1}{2}\left(19 - \frac{21r^2}{a^2}\right)\operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right)\right]\cos\theta, \\
\sigma_{zz} &= \frac{4Mr}{\pi a^4}\left[1 + \left(2 - \frac{3r^2}{a^2}\right)\operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right)\right]\cos\theta
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
\sigma_{r\theta} &= \frac{4Mr}{7\pi a^4}\left[1 - \operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right)\right]\sin\theta, \\
\sigma_{rz} &= -\frac{4M}{7\pi a^4}\left[\left(\frac{h}{2} - z\right) - \frac{1}{\omega}\left(22 - \frac{63r^2}{a^2}\right)\operatorname{sech}\frac{\omega h}{2}\sinh\omega\left(\frac{h}{2} - z\right)\right]\cos\theta, \\
\sigma_{\theta z} &= \frac{4M}{7\pi a^4}\left[\left(\frac{h}{2} - z\right) - \frac{1}{\omega}\left(22 - \frac{21r^2}{a^2}\right)\operatorname{sech}\frac{\omega h}{2}\sinh\omega\left(\frac{h}{2} - z\right)\right]\sin\theta.
\end{aligned} \tag{45}$$

The maximum values of σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz} and $\sigma_{\theta z}$ occur at the bonded ends $z = 0$ and $z = h$, where

$$\sigma = \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \frac{12Mr}{\pi a^4}\left(1 - \frac{r^2}{a^2}\right)\cos\theta, \tag{46}$$

$$\begin{aligned}
\sigma_{rz} &= \pm\frac{2Mh}{7\pi a^4}\left[1 - \frac{2}{\omega h}\left(22 - \frac{63r^2}{a^2}\right)\tanh\frac{\omega h}{2}\right]\cos\theta \\
\sigma_{\theta z} &= \mp\frac{2Mh}{7\pi a^4}\left[1 - \frac{2}{\omega h}\left(22 - \frac{21r^2}{a^2}\right)\tanh\frac{\omega h}{2}\right]\sin\theta
\end{aligned} \tag{47}$$

whilst $\sigma_{r\theta}$ has its maximum value on the central plane $z = h/2$, where

$$\sigma_{r\theta} = \frac{4Mr}{7\pi a^4}\left(1 - \operatorname{sech}\frac{\omega h}{2}\right)\sin\theta. \tag{48}$$

It is clear that the maximum value, $\sigma^{\max}$, of the components in Eqs. (46) occurs at $r = a/\sqrt{3}$ when $\theta = 0$ and is given by

$$\sigma^{\max} = \frac{8M}{\sqrt{3}\pi a^3}. \tag{49}$$

The component σ_{rz} in Eq. $(47)_1$ apparently reaches its maximum value, $\sigma_{rz}^{\max}$, of

$$\sigma_{rz}^{\max} = \frac{2Mh}{7\pi a^4}\left(1 + \frac{82}{\omega h}\tanh\frac{\omega h}{2}\right), \tag{50}$$

when $r = a$ and $\theta = 0$. Using Eq. (33), the maximum values in Eqs. (49) and (50) can be written in terms of the angular rotation ϕ as

$$\sigma^{\max} = \frac{7\mu\phi a}{\sqrt{3}h\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right)}, \tag{51}$$

$$\sigma_{rz}^{\max} = \frac{\mu\phi}{4}\frac{\left(1 + \frac{82}{\omega h}\tanh\frac{\omega h}{2}\right)}{\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right)}. \tag{52}$$

Again, if $h^2/a^2 < 5/42$, these can be closely approximated by

$$\sigma^{\max} = \frac{\mu\phi}{\sqrt{3}}\left(\frac{a}{h}\right)^3\left[1 + \frac{42}{5}\left(\frac{h}{a}\right)^2\right], \tag{53}$$

$$\sigma_{rz}^{\max} = \frac{3\mu\phi}{2}\left(\frac{a}{h}\right)^2\left[1 + \frac{41}{30}\left(\frac{h}{a}\right)^2\right], \tag{54}$$

when h/a is very small, the value (54) agrees with that suggested by Gent and Meinecke (1970, p. 52). However, it should be noted that in fact this theoretically predicted maximum value of the shear stress component, σ_{rz} (and similarly for $\sigma_{r\theta}$) at $r = a$ does not physically exist. Its determination through Eqs. $(45)_1$, (47), (50), (52), and (54) cannot apply actually at $r = a$, as both σ_{rz} and $\sigma_{r\theta}$ must decay rapidly to zero very near to this surface.

Further, Gent and Lindley (1958) postulated that bonded rubber cylinders subjected to a state of hydrostatic tension will 'fail' due to internal rupture when the magnitude of the tension reaches a critical value of between $0.783E$ and $0.833E$ $(= 5\mu/2)$. More recently Gent and Meinecke (1970) quoted $9\mu/4$, whilst Muhr (1992) suggested $5\mu/2$.

If the Gent and Meinecke criterion is adopted, it is found from Eq. (53) that the critical value, ϕ^*, of the angular rotation ϕ at which the rubber will fail is

$$\phi^* = \frac{9\sqrt{3}}{4}\left(\frac{h}{a}\right)^3\frac{1}{\left[1 + \frac{42}{5}\left(\frac{h}{a}\right)^2\right]}. \tag{55}$$

This value agrees with that of Gent and Meinecke (1970, Table 3) only for very short blocks.

More generally, if σ^* denotes the chosen critical value of the hydrostatic stress, it follows from Eq. (51) that correspondingly

$$\phi^* = \frac{\sqrt{3}\sigma^*}{7\mu}\left(\frac{h}{a}\right)\left(1 - \frac{a}{\sqrt{21}h}\tanh\frac{\sqrt{21}h}{a}\right). \tag{56}$$

4.3. Deformed profile

The deformed shape of the block in simple bending can be deduced approximately from the expression (21) for U.

For small deflections, the radius of curvature, $R(r,z)$, in the plane of bending of a line within the rubber which was originally parallel to the z-axis is given by

$$\frac{1}{R} = \frac{\partial^2 U}{\partial z^2}. \tag{57}$$

Thus, from Eq. (21) with $F = 0$,

$$\frac{1}{R} = -\frac{4M}{7\mu\pi a^4}\left[1 + 20\left(1 - \frac{63}{20}\frac{r^2}{a^2}\right)\operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right)\right]. \tag{58}$$

Classical bending theory predicts that the curvature of the deformed axis of the block does not depend upon z. However it is clear, by putting $r = 0$ into Eq. (58), that in fact the deformed axis is concave towards the negative x sectors for all values of z and h, but that its radius of curvature, $R_0 = R(0,z)$, varies with z according to

$$\frac{1}{R_0} = -\frac{4M}{7\mu\pi a^4}\left[1 + 20\operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right)\right]. \tag{59}$$

Correspondingly, the radius of curvature, $R_a = R(a,z)$, of the curved lateral sides of the block after deformation is given by

$$\frac{1}{R_a} = \frac{4M}{7\mu\pi a^4}\left[43\operatorname{sech}\frac{\omega h}{2}\cosh\omega\left(\frac{h}{2} - z\right) - 1\right]. \tag{60}$$

The deformed outer profile in the plane of bending is therefore convex towards the negative x sectors whenever $43\cosh\omega[(h/2) - z] > \cosh(wh/2)$. This inequality holds at the bonded ends $z = 0$ and $z = h$ for all values of h, but it is only satisfied when $z = h/2$ if $h < 0.972a$. However, blocks for which h is greater than this critical height have kinks in the surface $r = a$ near the central plane, $z = h/2$, whose extents spread more and more towards the bonded ends as h is increased. For example, the outer profile of a block with $h = 2a$ is concave towards the sectors in which x is negative, except in the regions $0.205 < z/h < 0.795$ near the bonded ends where the profile will be convex.

Some idea of the deformed outer profile in the plane of bending, as predicted by the current analysis, can be drawn from the locus of the points having Cartesian coordinates $([r + u_r]\cos\theta/a, 0, [z + u_z]/a)$ with $0 \leqslant z \leqslant h$, $r = a$ and $\theta = 0$ and π. Here u_r and u_z (which are assumed in this classical linear elasticity analysis to be infinitesimally small) are given by Eqs. (5), (19)–(21), (26), (27) and (33).

The pronounced difference in shape of the exaggeratedly-deformed profile and that of the central axis, $r = 0$, for blocks having $h/a = 0.5$ and 2 is apparent in Fig. 2(a) and (b), respectively, where $M/\mu\pi a^2$ has been chosen so that the angle of rotation, ϕ, of the end $z = h$ is 10°. M and ϕ are related through the relation (35). Such profiles can be drawn in just a fraction of a second using a mathematical computer

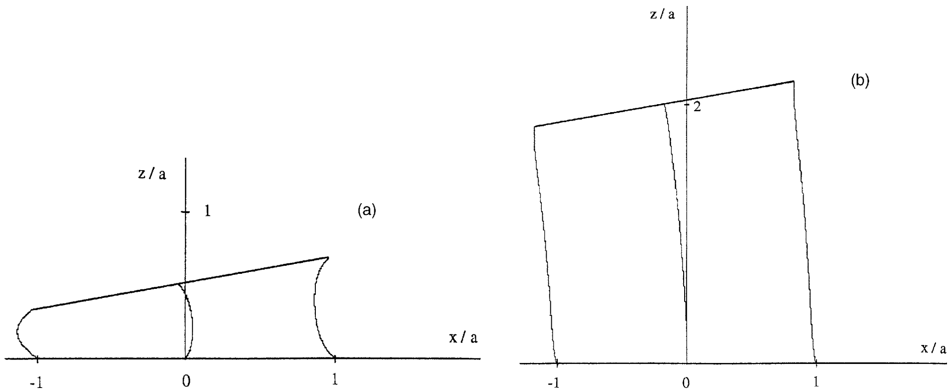

Fig. 2. Deformed profiles in simple bending with $\phi = 10°$ when (a) $h/a = 0.5$, (b) $h/a = 2$.

software package such as DERIVE on a PC. It is clear that the previously used assumption of parabolic profiles is indeed invalid (except perhaps for blocks of very small aspect ratio $h/a \ll 0.972$).

It is interesting to observe that, if a critical value of $\sigma^* = 5E/6$ $(= 5\mu/2)$ is chosen for the block depicted in Fig. 2(a), Eq. (56) yields $\phi^* \approx 10.14°$. Consequently angular strains greater than those creating the deformation in Fig. 2(a) would be likely to lead to internal failure of the rubber block originating at $r = a/\sqrt{3}$ on the two bonded ends where they are subject to tensile stress (in the positive x sectors).

5. Cantilever loading

The deformation of the block under "cantilever loading" can be studied by putting $M = Fh$ throughout the analysis in Section 3.

It then follows from Eqs. (28) and (29) that the angular rotation, ϕ, and the lateral deflection, d_b, due to bending alone, of the bonded end $z = h$ are given by

$$\phi = \frac{4Fh^2}{7\mu\pi a^4}\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right), \tag{61}$$

$$d_b = \frac{Fh}{147\mu\pi a^2}\left[1 + 28\left(\frac{h}{a}\right)^2 - \omega h \coth \omega h\right] - kh. \tag{62}$$

When h is large, these expressions yield

$$\phi \approx \frac{Fh^2}{2EI_e}, \quad d_b \approx \frac{Fh^3}{6EI_e} - kh, \tag{63}$$

where I_e is given by Eq. (39). The classical theory of bending predicts that for cantilevers $\phi = Fh^2/2EI$ and $d_b = Fh^3/3EI$, so that $d_b = 2h\phi/3$. The value of the constant k in the limit of large h can be found from Eqs. (63) to comply with this. However if this relationship is enforced for all values of h between the expressions (61) and (62) for ϕ and d_b, as with the corresponding relation in the simple bending situation in Section 4.1, the appropriate value of k can be found so that

$$d_b = \frac{8Fh^3}{21\mu\pi a^4}\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right). \tag{64}$$

Finally, when the deflection d_s due to the shear, given by Eq. (31), is superposed onto this, the total deflection, d, under cantilever loading can be expressed as

$$d = \frac{Fh}{\mu\pi a^2}\left[1 + \frac{8}{21}\left(\frac{h}{a}\right)^2\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right)\right]. \tag{65}$$

An approximation to the shape of the deformed profile can be deduced from the representations (19) and (21) for α and U with the appropriate value for k, if desired.

6. Apparent shear

When one end is displaced parallel to the other in its own plane the block experiences an "apparent shear" displacement. This occurs when $M = Fh/2$ in the general analysis of Section 3. But alternatively the deflection in apparent shear can be equivalently derived very conveniently by regarding it as the superposition of that in the cantilever loading of Section 5 when $M = Fh$ with that in the simple bending of Section 4 when $M = -Fh/2$.

6.1. Angular rotation and lateral deflection

It is clear that when $M = Fh/2$ in Eqs. (28) and (29)

$$\phi = 0 \tag{66}$$

as expected, and

$$d_b = \frac{Fh}{147\mu\pi a^2}\left[1 + 7\left(\frac{h}{a}\right)^2 - \frac{\omega h}{2}\coth\frac{\omega h}{2}\right] - kh. \tag{67}$$

On the other hand, by superposition of Eqs. (64) and (34) with $M = -Fh/2$

$$d_b = \frac{2Fh^3}{21\mu\pi a^4}\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right). \tag{68}$$

The equivalence of Eqs. (67) and (68) enables the constant k, which is needed in Eq. (21) for describing the deformed profile, to be determined.

The deflection d_s from Eq. (31) due to the simple shear alone must be added to the expression (68) to give a representation for the total deflection, d, of the bonded end $z = h$ in apparent shear as

$$d = \frac{Fh}{\mu\pi a^2}\left[1 + \frac{2}{21}\left(\frac{h}{a}\right)^2\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right)\right]. \tag{69}$$

From this the ratio, μ/μ_a, of the true shear modulus, μ, to the apparent shear modulus, μ_a, under these loading conditions can be written in the form

$$\frac{\mu}{\mu_a} = 1 + \frac{2}{21}\left(\frac{h}{a}\right)^2\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right). \tag{70}$$

When h is large, Eq. (69) yields

$$d \approx \frac{Fh^3}{12EI_e}, \tag{71}$$

with I_e given by Eq. (39), which is analogous with the form in the classical theory of beams.

On the other hand, if h/a is small, a series expansion shows that when $h^2/a^2 < 5/42$ the expression (70) is closely approximated (with an error of magnitude less than 0.003%) by

$$\left(\frac{\mu}{\mu_a}\right)^{\text{approx}} = 1 + \frac{h^2}{36k_c^2\left[1.4 + \frac{1}{6}\left(\frac{a}{h}\right)^2\right]}, \tag{72}$$

where $k_c (= a/2)$ is the radius of gyration of the circular cross-section.

Rivlin and Saunders (1949) presented both theoretical and experimental results for the ratio μ/μ_a. Their suggested approximation, $(\mu/\mu_a)^{RS}$ for comparison with the experiments was simply

$$\left(\frac{\mu}{\mu_a}\right)^{RS} = 1 + \frac{h^2}{36k_c^2}. \tag{73}$$

However, Gent and Meinecke (1970) observed that this is only adequate when the bending deflection is appreciable, i.e. when h is large, and instead proposed the approximate ratio, $(\mu/\mu_a)^{GM}$ given by

$$\left(\frac{\mu}{\mu_a}\right)^{GM} = 1 + \frac{h^2}{36k_c^2\left[1 + \frac{1}{6}\left(\frac{a}{h}\right)^2\right]}. \tag{74}$$

The relation (72) for the ratio provides a better estimate to the exact representation (70) than those of Eqs. (73) and (74). Comparisons of the values obtained for the ratio using the expressions (70), (73) and (74) with the experimental results, $(\mu/\mu_a)^{exp}$, given by Rivlin and Saunders (1949) are displayed in Table 1 and Fig. 3.

6.2. Stresses

The normal stress components in the block under apparent shear can be deduced from Eqs. (13), (16), $(8)_2$, (15) and (19), and it is found that the maximum values of these occur at the bonded ends $z = 0$ and h where

$$\sigma = \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \pm \frac{6F}{\pi a^2}\left(\frac{h}{a}\right)\left(\frac{r}{a}\right)\left(1 - \frac{r^2}{a^2}\right)\cos\theta. \tag{75}$$

Their maximum magnitude, σ^{max}, clearly occurs at $\theta = 0$ when $r = a/\sqrt{3}$ and is given by

Table 1
Ratio of the true shear modulus to the apparent shear modulus

h/a	0	0.25	0.75	1.5	2	3	4
$(\mu/\mu_a)^{exp}$	–	0.9898	1.117	1.161	1.35	1.787	2.62
μ/μ_a	1.0	1.002	1.038	1.183	1.339	1.795	2.44
$(\mu/\mu_a)^{GM}$	1.0	1.002	1.048	1.233	1.427	1.982	2.76
$(\mu/\mu_a)^{RS}$	1.0	1.007	1.063	1.25	1.444	2.0	2.78

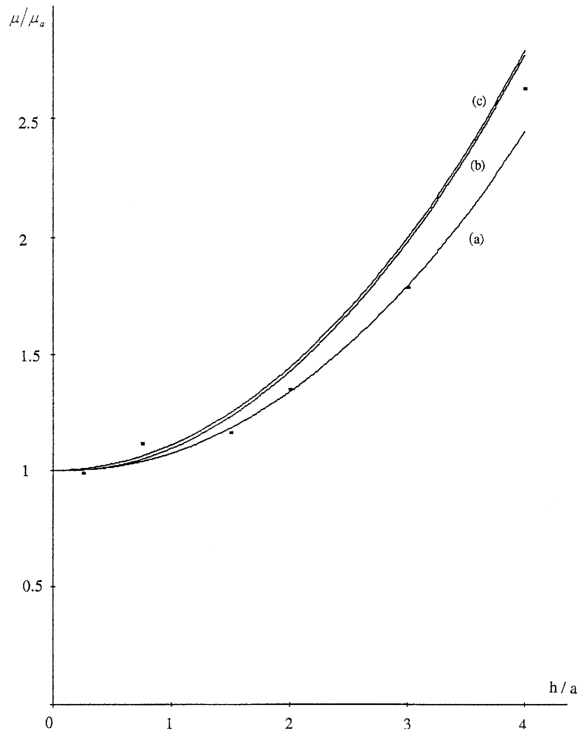

Fig. 3. Variation with h/a of (a) μ/μ_a, (b) $(\mu/\mu_a)^{GM}$, (c) $(\mu/\mu_a)^{RS}$, (■) $(\mu/\mu_a)^{exp}$.

$$\sigma^{\max} = \frac{4F}{\sqrt{3}\pi a^2}\left(\frac{h}{a}\right). \tag{76}$$

If the critical value at which rupture occurs is chosen to be $\sigma^* = 5\mu/2$, it is found, using the relationship (69) connecting F and d, that the critical value, $(d/a)^*$, of the scaled lateral deflection of the bonded end $z = h$ is

$$\left(\frac{d}{a}\right)^* = \frac{5\sqrt{3}}{8}\left[1 + \frac{2}{21}\left(\frac{h}{a}\right)^2\left(1 - \frac{2}{\omega h}\tanh\frac{\omega h}{2}\right)\right]. \tag{77}$$

6.3. Deformed profile

A general idea of the deformed shape of the block in apparent shear can be obtained from Eq. (61), as in Section 4.3, with U given by Eqs. (21) and (30) when $M = Fh/2$ and k obtained from Eqs. (67) and (68). It is found that the radius of curvature R is given by

$$\frac{1}{R} = -\frac{2Fh}{7\mu\pi a^4}\left[20\left(1 - \frac{63}{20}\frac{r^2}{a^2}\right)\operatorname{cosech}\frac{\omega h}{2}\sinh\omega\left(\frac{h}{2} - z\right) + \left(1 - \frac{2z}{h}\right)\right]. \tag{78}$$

This is asymmetrical about the plane $z = h/2$, and is actually zero when $z = h/2$. It is clear, by putting $r = 0$, that for all values of h the deformed axis of the block is concave towards the negative x sectors for $0 \leqslant z < h/2$ but is convex for $h/2 < z \leqslant h$. However, the radius of curvature, R_a, of the deformed curved lateral sides can be found from

$$\frac{1}{R_a} = \frac{2Fh}{7\mu\pi a^4}\left[43\operatorname{cosech}\frac{\omega h}{2}\sinh\omega\left(\frac{h}{2} - z\right) - \left(1 - \frac{2z}{h}\right)\right]. \tag{79}$$

The deformed outer profile is therefore convex towards the negative x sectors whenever $43\sinh\omega[(h/2) - z] > (1 - 2z/h)\sinh(\omega h/2)$. This inequality holds for all values of h at the bonded end $z = 0$. Conversely, at the end $z = h$ the deformed outer profile is concave towards the negative x sectors for all values of h. Furthermore, the lateral surface slightly above the central plane $z = h/2$ will be concave towards the negative x sectors if $h < 1.373a$ and convex otherwise, whilst slightly below $z = h/2$ it will be convex if $h < 1.373a$ and concave otherwise.

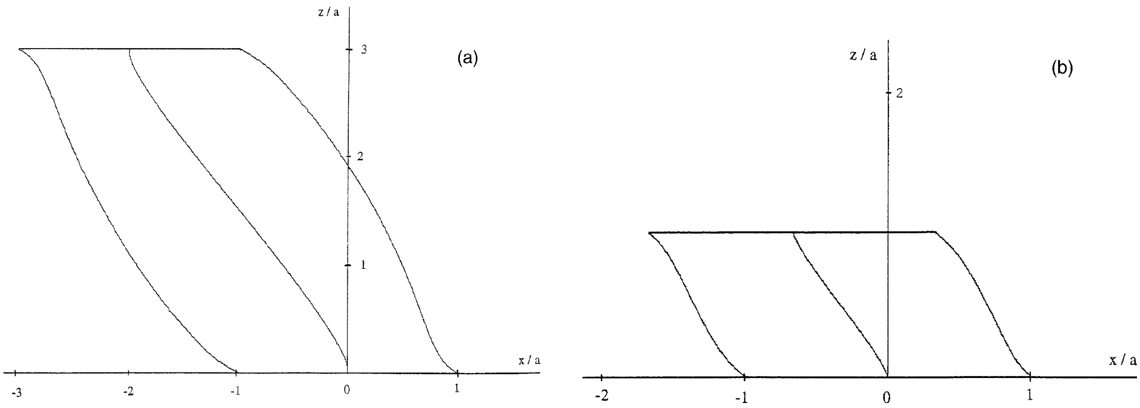

Fig. 4. Deformed profiles in apparent shear when (a) $d = 2a$, $h/a = 3$ (b) $d = 2a/3$, $h/a = 1$.

Some idea of the exact deformed outer profile and central axis, as predicted by the current analysis, can be drawn as outlined in Section 4.3. Rivlin and Saunders (1949, Fig. 1(c)) presented a photograph of their experimental work with a deformed cylinder in a shear mounting, in which it appears that approximately the lateral deflection is twice the radius for a cylinder whose length is three times the radius. For direct comparison the curves in Fig. 4(a) have therefore been calculated with $F/\mu\pi a$ chosen so that $d = 2a$ and with $h/a = 3$. F and d are connected according to the relation (69). The profile can be seen to agree very closely with that of Rivlin and Saunders (1949, Fig. 1(c)). It is noteworthy that with these dimensions Rivlin and Saunders were actually very close to attaining internal rupture, as when $h/a = 3$ the relation (77) gives $(d/a)^* = 1.94$. For comparison, the shape in Fig. 4(b) depicts a block with $d = 2a/3$ and $h/a = 1$, for which $(d/a)^* = 1.16$.

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